PG – 620

II Semester M.Sc. Degree Examination, July 2017
(RNS Repeaters)
(2011 – 12 & Onwards)
MATHEMATICS
M-203 : Functional Analysis

Time : 3 Hours Max. Marks : 80

Instructions : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Define a Banach Space. Show that l_2^n is a Banach Space.

   b) If N is a normed linear space and M is a closed subspace of N then show that the quotient space $\frac{N}{M}$ is a normed linear space. Further show that $\frac{N}{M}$ is complete when N is complete. **(8+8)**

2. a) Let T be a linear transformation of a normed linear space N into a normed linear space N'. Then show that T is bounded if and only if T is continuous.

   b) Let M be a closed subspace of a normed linear space N. Then show that there is a linear continuous transformation $T : N \to N/M$ with $\|T\| \le 1$.

   c) If $S : N \to N'$ is a continuous linear transformation and if M is the null space of S then prove that there is a linear transformation $S' : \frac{N}{M} \to N'$ such that $\|S\| = \|S'\|$. **(7+4+5)**

3. a) State and prove Hahn-Banach theorem for a real normed linear space.

   b) If N is a normed linear space and $X_0 \in N$ such that $x_0 \ne 0$, then prove that there is a functional $f_0 \in N^*$ such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$. **(10+6)**

P.T.O.

4. a) Show that there is a natural embedding of N into N** obtained by isometric isomorphism $\phi : N \to N^{**}$ defined by $\phi(x) = F_x$ where $F_x(f) = f(x),\ \forall\, f \in N^*, x \in N$.

   b) State and prove uniform boundedness theorem. **(8+8)**

5. a) State and prove :

   i) Schwartz inequality

   ii) Bessel's inequality.

   b) Let S_1 and S_2 be any subsets of a Hilbert space H. Then prove that :

   i) $S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \subseteq S_1^{\perp}$

   ii) $S^{\perp}$ is a closed linear subspace of H.

   c) Show that translation on a Hilbert space is continuous. **(7+6+3)**

6. a) Let M and N be closed linear subspaces of a Hilbert space H such that $M \perp N$. Then prove that M + N is a closed linear subspace of H.

   b) Show that every non zero Hilbert space contains a complete orthonormas set.

   c) Define adjoint of an operator. With usual notations prove the following properties of an adjoint operation.

   i) $(T_1 + T_2)^* = T_1^* + T_2^*$

   ii) $(\alpha T)^* = \overline{\alpha} T^*$

   iii) $(T_1 T_2)^* = T_2^* \, T_1^*$ **(5+6+5)**

7. a) If T is an operator on a Hilbert space H then prove that

   i) $T = 0$ if and only if $\langle Tx, y\rangle = 0$

   ii) $T = 0$ if and only if $\langle Tx, x\rangle = 0,\ \forall x, y \in H$.

   b) Define self adjoint operator. Show that self adjoint operators have real eigen values.

   c) If N_1 and N_2 are normal operators on a Hilbert space H with the property that $N_1 N_2^* = N_2^* N_1$ and $N_1^* N_2 = N_2 N_1^*$ then prove that $N_1 + N_2$ and $N_1\, N_2$ are normal. **(5+4+7)**

8. a) Define unitary operator. Show that the following are equivalent for an operator T on a Hilbert space H.

i) $T^*T = I$

ii) $\langle Tx, Ty \rangle = \langle x, y \rangle, \ \forall x, y \in H$

iii) $\|Tx\| = \|x\|, \ \forall x \in H$.

b) Show that an operator T on a Hilbert space H is unitary if and only if it is an isometric isomorphism of H onto itself.

c) Define spectrum of a linear operator T on H. Show that spectrum of T is non empty. **(6+7+3)**